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DYNAMIC RESPONSE OF LAMINATED PLATES TO RANDOM LOADING

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Abstract—The subject of this paper is the response of laminated plates in cylindrical bending to random dynamic loading. A random loading is characterized by a weakly stationary stochastic process.

The formulae for correlation function (and variance) of vertical displacement of a plate are obtained and the numerical results are provided for the assumed form of the correlation function of a random loading. The effect of the fiber orientation and the correlation parameter of the external loading on the mean-square value of vertical displacement are shown graphically. The results are compared with those obtained using the classical Kirchhoff and Mindlin theory for homogeneous plates.

1. INTRODUCTION

Because of practical importance of composite materials in various branches of engineering an increasing amount of attention has been recently devoted to different problems associated with the mechanical behaviour of composite laminated plates. As a result, various approaches have been developed to establish the appropriate theory of laminated plates. For example, Lekhnitskii[1] presents the theory of symmetrical laminates based on the Kirchhoff hypothesis. Reissner and Stavsky[2] included the effects of bending-extensional coupling in unsymmetrical laminates. For the dynamic behaviour of laminated plates, Mindlin's theory[3] for homogeneous isotropic plates has been extended to laminated plates by Yang, Norris and Stavsky[4] and by Whitney and Pagano[5]. Therefore, these studies include the effects of transverse shear deformation and rotary inertia. Since the number of layers in a laminate is often large, it is natural to use a continuum type theory; such a theory has been proposed by Sun *et al.*[6].

At the present time there exists a large number of papers solving various problems for laminated composite elastic plates (static bending, eigenfrequences, wave propagation etc.). However, up to now the problems of vibrations of such plates due to time-varying excitation have not been investigated extensively enough. The recent papers[7, 8] by Sun and Whitney seem to be the most representative; they provide an analysis of forced vibrations of laminated plates subjected to deterministic dynamic loading.

The subject of this paper is the response of laminated plates in cylindrical bending to random dynamic loading. The analysis makes use of the theory of laminated plates which includes the effect of transverse shear deformation and rotary inertia. A random loading is characterized by a weakly stationary stochastic process. The general formulae for correlation function (and variance) of vertical displacement of a plate are obtained and the numerical results are provided for the assumed (exponential) form of correlation function of random loading. Assuming that the random loading is gaussian the results of calculations of the average number of crossings of a given level by the random displacement field are also provided. The results are compared with these obtained using the classical Kirchhoff and Mindlin theory for elastic homogeneous plates.

2. STATEMENT OF THE PROBLEM

Consider a plate of constant thickness h composed of finite number of thin layers of elastic, homogeneous and anisotropic material bonded together. The origin of a Cartesian coordinate system is located within the central plane x-y with the z-axis being normal to this plane. The material of each layer is assumed to possess a plane of elastic symmetry parallel to the x-y

plane. It is assumed that the plate is of infinite extent in the y-direction and its width is equal to a, i.e. $x \in [0, a]$. In general, we assume that the plate surfaces: $z = \pm (h/2)$ are subjected to the following surface loads (independent of y):

$$
\begin{aligned}
p_x &= \tau_{xz}\left(x, \frac{h}{2}, t\right) - \tau_{xz}\left(x, -\frac{h}{2}, t\right),\\
p_y &= \tau_{yz}\left(x, \frac{h}{2}, t\right) - \tau_{yz}\left(x, -\frac{h}{2}, t\right),\\
p &= \tau_{zz}\left(x, \frac{h}{2}, t\right) - \tau_{zz}\left(x, -\frac{h}{2}, t\right),\\
m_x &= \frac{n}{2}\left[\tau_{xz}\left(x, \frac{h}{2}, t\right) + \tau_{xz}\left(x, -\frac{h}{2}, t\right)\right],\\
m_y &= \frac{h}{2}\left[\tau_{yz}\left(x, \frac{h}{2}, t\right) + \tau_{yz}\left(x, \frac{h}{2}, t\right)\right],
\end{aligned}
\tag{2.1}
$$

where τ_{xz}, τ_{yz} and τ_{zz} are the tangential and normal stresses acting on the surfaces, respectively. In what follows, it will be assumed that $p = p(x, t)$ is a stationary stochastic process with respect to time t.

Using the theory of laminated plates presented in [4] and [5, 7] the displacement field of a plate is assumed in the following form

$$
\begin{aligned}
u(x, y, t) &= u^{\circ}(x, y, t) + z\psi_x(x, y, t)\\
v(x, y, t) &= v^{\circ}(x, y, t) + z\psi_y(x, y, t)\\
w(x, y, t) &= w(x, y, t)
\end{aligned}
\tag{2.2}
$$

where u, v, w are displacement components in the x, y and z direction, respectively.

The displacement equations including the damping effects are of the form (see [7])

$$
\begin{aligned}
&A_{11}u^0_{,xx} + A_{16}v^0_{,xx} + B_{11}\psi_{x,xx} + B_{16}\psi_{y,xx} + p_x = P\ddot{u}^0 + R\ddot{\psi}_x + 2\epsilon p\dot{u}^0 + 2\epsilon R\dot{\psi}_x,\\
&A_{16}u^0_{,xx} + A_{66}v^0_{,xx} + B_{16}\psi_{x,xx} + B_{66}\psi_{y,xx} + p_y = P\ddot{v}^0 + R\ddot{\psi}_y + 2\epsilon P\dot{v}^0 + 2\epsilon R\dot{\psi}_y,\\
&k[A_{55}(\psi_{x,x} + w_{,xx}) + A_{45}\psi_{y,x}] + p = P\ddot{w} + 2\epsilon P\dot{w}\\
&B_{11}u^0_{,xx} + B_{16}v^0_{,xx} + D_{11}\psi_{x,xx} + D_{16}\psi_{y,xx} - k[A_{55}(\psi_x + w_{,x})\\
&\quad + A_{45}\psi_y] + m_x = R\ddot{u}^0 + I\ddot{\psi}_x + 2\epsilon R\dot{u}^0 + 2\epsilon I\dot{\psi}_x,\\
&B_{16}u^0_{,xx} + B_{66}v^0_{,xx} + D_{16}\psi_{x,xx} + D_{66}\psi_{y,xx} - k[A_{45}(\psi_x + w_{,x})\\
&\quad + A_{44}\psi_y] + m_y = R\ddot{v}^0 + I\ddot{\psi}_y + 2\epsilon R\dot{v}^0 + 2\epsilon I\dot{\psi}_y,
\end{aligned}
\tag{2.3}
$$

where

$$
(P, R, I) = \int_{-h/2}^{+h/2} \rho(1, z, z^2)\,\mathrm{d}z, \quad \rho\text{–mass density}; \tag{2.4}
$$

the stiffnesses of the laminate A_{ij}, B_{ij}, D_{ij} are:

$$
\begin{aligned}
(A_{ij}, B_{ij}, D_{ij}) &= \int_{-h/2}^{+h/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z, \qquad i,j = 1, 2, 6,\\
A_{ij} &= \int_{-h/2}^{+h/2} C_{ij}\,\mathrm{d}z, \quad i, j = 4, 5,\\
Q_{i\alpha} &= C_{i\alpha} - \frac{C_{i3}}{C_{33}} C_{3\alpha}, \quad i = 1, 2, 3, 6, \qquad \alpha = 1, 2, 6
\end{aligned}
\tag{2.5}
$$

and the anisotropic transverse shear stifnesses C_{ij} are defined by the constitutive relations for any layer

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{36} \\ C_{16} & C_{26} & C_{36} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \end{bmatrix} \tag{2.6}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{bmatrix} \epsilon_{yz} \\ \epsilon_{xz} \end{bmatrix} \tag{2.7}$$

The parameter k is introduced as a shear correction factor in the same manner as in Mindlin's theory for homogeneous isotropic plates. The possible values of k are, however, not uniquely defined; the range of values of k depending on the material properties of a laminate have been calculated numerically in [4] for a two-layer isotropic plate. There are some indications in the existing literature, that the value $(\pi^2/12)$ yields reasonably accurate results for the frequency spectrum of the first flexural mode for a broad variety of composite laminates. The parameter ϵ introduced into the governing equations is a damping coefficient.

The eqns (2.3) should be solved with the appropriate initial conditions and the boundary conditions posed on the edges of the plate. In general, the boundary conditions may be written as

$$B_i(u^0, v^0, w, \psi_x, \psi_y) = f_i(t), i = 1, 2, \ldots, 10 \tag{2.8}$$

where $f_i(t)$ are given functions of time defined on the edges $x = 0$, $x = a$.

We wish to determine information on the dependence of a random response of a laminated plate upon the correlation radius of external loading and upon the different variants of distribution of plate components. To do this, we shall join the standard method of separation of variables with a stochastic reasoning of statistical dynamics (see [9, 10]).

3. SOLUTION

In what follows we assume that a random pressure $p = p(x, t)$ is applied on the top surface of the plate (in such a case: $p_x = p_y = m_x = m_y = 0$) and the boundary conditions (2.8) are homogeneous, that is $f_i(t) = 0$ (e.g. the plate is simply supported along the edges $x = 0$, $x = a$).

A solution of eqns (2.3) is looked for in the form:

$$\begin{aligned} u^0(x, t) &= \sum_{n=1}^{\infty} U_n(x)T_n(t), \\ v^0(x, t) &= \sum_{n=1}^{\infty} V_n(x)T_n(t), \\ w(x, t) &= \sum_{n=1}^{\infty} W_n(x)T_n(t), \\ \psi_x(x, t) &= \sum_{n=1}^{\infty} \psi_{xn}(x)T_n(t), \\ \psi_y(x, t) &= \sum_{n=1}^{\infty} \psi_{yn}(x)T_n(t), \end{aligned} \tag{3.1}$$

where U_n, V_n, W_n, ψ_{xn}, ψ_{yn} are the eigenfunctions of the plate (determined by homogeneous equations of motion and homogeneous boundary conditions). $T_n(t)$ are the functions to be found; in our case $T_n(t)$ are unknown stochastic processes.

Representing the external loading $p(x, t)$ in the form

$$p(x, t) = \sum_{n=1}^{\infty} Q_n(t) P W_n(x) \tag{3.2}$$

and substituting (3.1) and (3.2) into governing eqns (2.3) one obtains the following stochastic equation for $T_n(t)$:

$$\ddot{T}_n(t) + 2\epsilon_n \dot{T}_n(t) + \omega_n{}^2 T(t) = Q_n(t) \tag{3.3}$$

where

$$Q_n(t) = \frac{1}{J_{nn}} \int_0^a p(x, t) W_n(x)\, dx, \tag{3.4}$$

$$J_{nn} = \int_0^a [P(U_n{}^2 + V_n{}^2 + W_n{}^2) + 2R(\psi_{xn} U_n + \psi_{yn} V_n) + I(\psi_{xn}^2 + \psi_{yn}^2)]\, dx. \tag{3.5}$$

In eqn (3.3) ω_n denotes the natural frequency associated with the principal modes U_n, V_n, W_n, ψ_{xn}, ψ_{yn} and ϵ_n is the corresponding attenuation coefficient.

Formula (3.4) makes it easy to obtain the correlation function (and spectral density) of $Q_n(t)$. Having these characteristics one gets the following formula for the correlation function of the processes $T_n(t)$:

$$K_{T_nT_m}(\tau) = \int_{-\infty}^{+\infty} \frac{S_{Q_nQ_m}(\omega)}{L_n(i\omega) L_m(-i\omega)} e^{i\omega\tau}\, d\omega \tag{3.6}$$

where $\tau = t_2 - t_1$ and

$$L_n(i\omega) = \omega_n{}^2 + 2i\epsilon_n\omega - \omega^2, \tag{3.7}$$

$$L_m(-i\omega) = \omega_m{}^2 - 2i\epsilon_m - \omega^2; \tag{3.8}$$

$S_{Q_nQ_m}(\omega)$ is the mutual spectral density of $Q_n(t)$ and $Q_m(t)$.

Temporal correlation functions of the displacement of the plate are obtained by averaging of the products of solutions (3.1) taken in two different time instants t_1 and t_2. For instance, the correlation function of vertical displacement at the fixed point x is

$$K_W(x, \tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_n(x) W_m(x) K_{T_nT_m}(\tau). \tag{3.9}$$

The mean-square value of the vertical displacement in the center of the plate is

$$\langle w^2 \rangle = K_w\left(\frac{a}{2}, 0\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_n\left(\frac{a}{2}\right) W_m\left(\frac{a}{2}\right) K_{T_nT_m}(0). \tag{3.10}$$

4. PARTICULAR CASE

To obtain quantitative results we assume here that the plate is simply supported along the edges $x = 0$, $x = a$ and of symmetric construction (coefficients B_{ij} in eqns (2.3 vanish) in which each ply is orthotropic with respect to the x-y axis and the density ρ of each layer is the same; orthotropy implies that $A_{16} = A_{26} = D_{16} = D_{26} = 0$. It is assumed furthermore that the external loading is of the form

$$p(x, t) = p_0 \xi(t) \tag{4.1}$$

where p_0 is a constant and $\xi(t)$ is a given stationary stochastic process characterizing the time-varying loading of the plate.

Under the above assumptions

$$W_n(x) = A_n h \sin\frac{n\pi x}{a},$$

$$J_{nn} = A_n^2 \rho h^3 a,$$

$$A_n = \left(\frac{2}{aPh^2}\right)^{1/2}. \tag{4.2}$$

In the case of cylindrical bending, the vibration frequences ω_n are (see [5])

$$\omega_n^2 = \frac{6}{Pa^2h^2}\left\{D_{11}n^2\pi^2 + kA_{55}\left(a^2 + \frac{n^2\pi^2h^2}{12}\right)\right.$$
$$\left. + kA_{55}\left[\left(\frac{D_{11}n^2\pi^2}{kA_{55}} + a^2 - \frac{n^2\pi^2h^2}{12}\right)^2 + \frac{n^2\pi^2a^2h^2}{3}\right]^{1/2}\right\}; \tag{4.3}$$

if one neglects rotary inertia, the above expression reduces to the following:

$$\omega_n = \bar{\omega}_n\left(1 - \frac{D_{11}n^2\pi^2}{D_{11}n^2\pi^2 + kA_{55}a^2}\right)^{1/2} \tag{4.4}$$

where

$$\bar{\omega}_n = \left(\frac{D_{11}n^4\pi^4}{Pa^4}\right)^{1/2}$$

denotes the frequency calculated from the classical Kirchhoff theory.

In the case under consideration formula (3.4) yields

$$Q_n(t) = \frac{p_0}{J_{nn}}\xi(t)\int_0^a A_n h \sin\frac{n\pi x}{a}\,dx = C_n\xi(t), \tag{4.5}$$

where constants C_n are

$$C_n = \frac{2ap_0hAn}{n\pi J_{nn}}, \quad n\text{—odd}$$

$$C_n = 0, \qquad n\text{—even}. \tag{4.6}$$

Formulae (3.9) and (3.10) take the form

$$K_w(x, \tau) = \sum_{n=1,3,5,\ldots}\sum_{m=1,3,5,\ldots} A_nA_mh^2 \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a} K_{T_nT_m}(\tau), \tag{4.7}$$

$$\langle w^2\rangle = \sum_{n=1,3,5,\ldots}\sum_{m=1,3,5,\ldots} A_nA_mh^2 \sin\frac{n\pi}{2}\sin\frac{m\pi}{2} K_{T_nT}(0), \tag{4.8}$$

where

$$K_{T_nT_m}(\tau) = C_nC_m\int_{-\infty}^{+\infty}\frac{S_\xi(\omega)}{L_n(i\omega)L_m(-i\omega)}e^{i\omega\tau}\,d\omega. \tag{4.9}$$

5. NUMERICAL RESULTS

Assuming the following spectral density of $\xi(t)$:

$$S_\xi(\omega) = \frac{\sigma^2\alpha}{\pi(\alpha^2 + \omega^2)} \tag{5.1}$$

Table 1.

Case number	Number of layers	Fiber orientation angle (to axis x)	Thickness of layers
1	1	0	h
2	1	$\pi/2$	h
3	2	$0, \pi/2$	$h/2\ h/2$
4	3	$0, \pi/2, 0$	$h/3\ h/3\ h/3$
5	3	$\pi/2, 0, \pi/2$	$h/3\ h/3\ h/3$
6	4	$0, \pi/2, \pi/2, 0$	$h/4\ h/4\ h/4\ h/4$
7	4	$\pi/2, 0, 0, \pi/2$	$h/4\ h/4\ h/4\ h/4$
8	5	$0, \pi/2, 0, \pi/2, 0$	$h/5\ h/5\ h/5\ h/5\ h/5$
9	5	$\pi/2, 0, \pi/2, 0, \pi/2$	$h/5\ h/5\ h/5\ h/5\ h/5$
10	1	0	h

10-classical plate theory

where σ^2 is the variance of the excitation and α–the correlation parameter, integral (4.9) was evaluated by use of the residue method and then the values of $\langle w^2 \rangle$ according to (4.8) were calculated. The assumed properties of individual ply are the same as in [7], that is

$$E_L = 20E_0, E_T = E_0, E_0 = 10^6, \nu_{LT} = \nu_{TL} = 0.25, G_{LT} = 0.6 \times 10^6, G_{TT} = 0.5 \times 10^6$$

where L and T denote the directions parallel and normal to the fibers, respectively, ν_{LT} is the Poisson ratio measuring transverse strain under uniaxial normal stress parallel to the fibers. The value of k was assumed $(\pi^2/12)$. Various cases of the fiber orientation and the number of layers are tabulated in Table 1. The case 1 and 10 correspond to the homogeneous (not laminated) plate according to the classical Kirchhoff and Mindlin theory, respectively.

The result of calculation of the mean-square of vertical displacement $\langle w^2 \rangle$ in the center of the plate is shown in Figs. 1 and 2 as a function of the correlation parameter $\beta = (\alpha/\omega_1)$. It is seen that the mean-square $\langle \omega^2 \rangle$ takes the smallest values for not laminated plates (for all practically reasonable values of β). Figures 1 and 2 reveal that the mean-square $\langle \omega^2 \rangle$ depends essentially on the fiber direction in the top and bottom layers (compare: cases 8 and 9, 6 and 7, 4 and 5). The number of layers also affects the mean-square significantly. Figure 3 illustrates the mean-square $\langle \omega^2 \rangle$ as a function of (a/h).

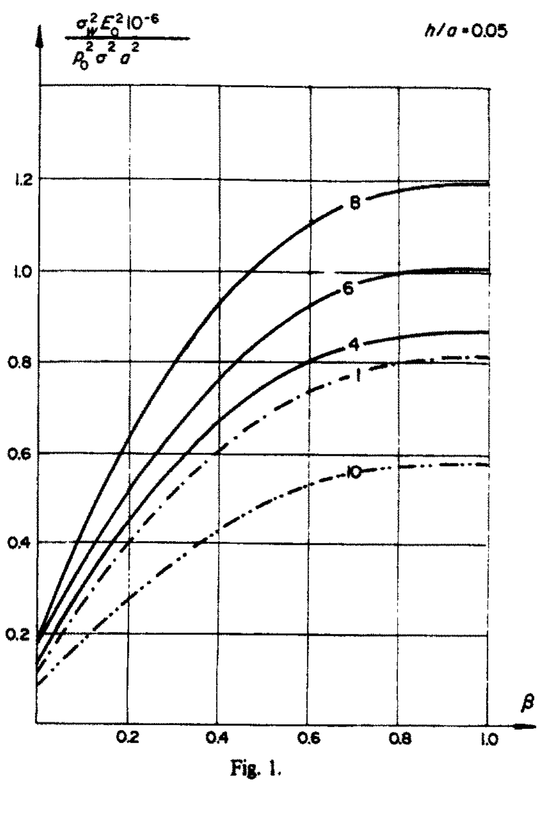

Fig. 1.

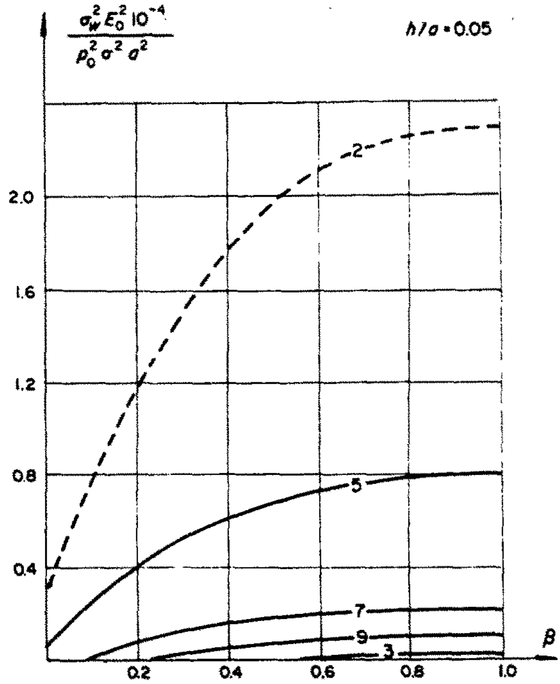

Fig. 2.

It was also of interest to see the dependence of the simplest reliability characteristics of the considered system upon the correlation parameter of the external loading. Assuming that the response process is gaussian the average number of its crossings $E[N_w(\zeta)]$ of a given level ζ in the unit time was calculated, according to well known formula[9]:

$$E[N_w(\zeta)] = \frac{1}{2}\frac{\sigma_{\dot{w}}}{\sigma_w}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right). \qquad 5.2)$$

In calculations the level ζ was assumed to be $1.5\, w_{max}$, where w_{max} is the maximum static displacement of the classical homogeneous plate. The results of this calculation are shown in

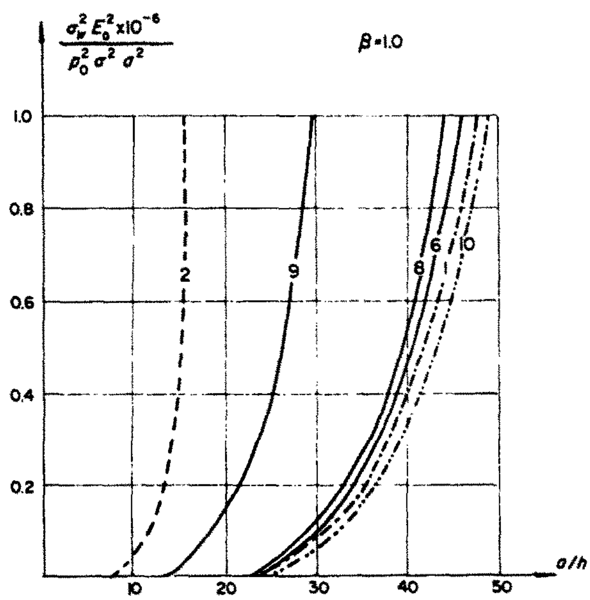

Fig. 3.

Fig. 4. It is easily observed that quantity (5.2) is strongly influenced by the correlation parameter occuring in the description of random loading, but the fiber orientation (the way of layering) does not appear to be essential.

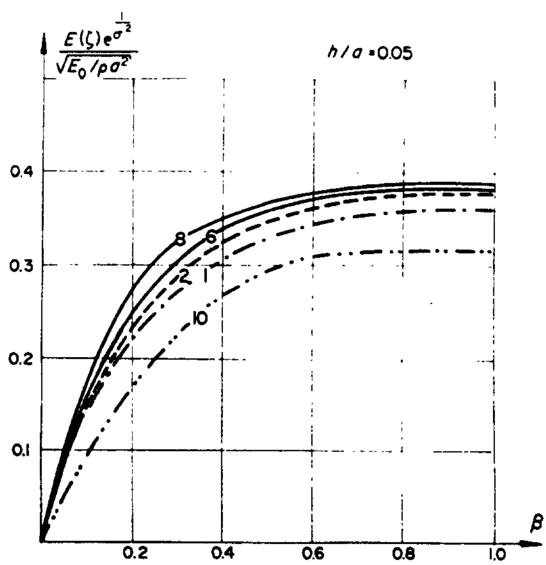

Fig. 4.

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